NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4311

PRANDTL NUMBER EFFECTS ON UNSTEADY FORCED-CONVECTION HEAT TRANSFER

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SUMMARY

An analysis is made for laminar forced-convection heat transfer on a flat plate with unsteady surface temperature. The deviation of the instantaneous heat-transfer rate from the quasi-steady value is computed. Results are obtained for Prandtl numbers in the range 0.01 to 10. The deviations from quasi-steady heat transfer increase markedly with increasing Prandtl number. The findings reported here should apply approximately in the entrance region of ducts and should also provide an upper bound on deviations from turbulent quasi-steady heat transfer.

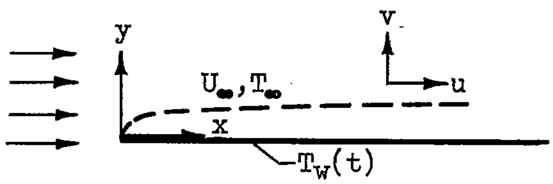
INTRODUCTION

It is often necessary to compute the forced-convection heat transfer from a surface whose temperature is changing with time. This problem is much simplified when it is supposed that the boundary layer passes through a succession of instantaneous steady states. Such a boundary layer is called quasi-steady. Under this assumption, the heat transfer is computed by instantaneous application of steady-state heat-transfer relations. The quasi-steady supposition is also invoked in heat-transfer experiments employing the transient technique, where the instantaneous measurements are used to determine steady-state coefficients.

In reality, there is always a difference between the actual instantaneous heat transfer and the quasi-steady value. The extent of the deviation depends upon the response characteristics of the boundary layer, as well as on the rapidity of the changes in surface temperature.

The aim of this analysis is to find the first- and second-order deviations of the actual instantaneous heat transfer from the quasi-steady value and to learn how these deviations depend upon the Prandtl number. The final results provide a rapid and accurate quantitative means for determining when a given set of surface temperature and free-stream velocity data lead to essentially quasi-steady heat transfer.

The system chosen for study is a flat plate alined parallel to a steady laminar flow as pictured in the following sketch:



The surface temperature T_w is spatially uniform but is permitted to take on arbitrary, but continuously differientable, variations with time. The free-stream temperature T_∞ is taken to be constant. Results are given here for fluids having Prandtl numbers between 0.01 and 10. This investigation constitutes an extension of previous work for $Pr = 0.72$ (air and other gases) reported in reference 1. Those primarily interested in results are invited to pass over the section on ANALYSIS. 4868

Readers interested in non-quasi-steady boundary layers are referred to the work of Moore and Ostrach (refs. 2 to 5), who studied the effects associated with time variations in free-stream velocity.

SYMBOLS

c_p	specific heat at constant pressure
F	Blasius velocity function
h	local heat-transfer coefficient, $q/\Delta T$
k	thermal conductivity
M_∞	free-stream Mach number
Pr	Prandtl number, $c_p\mu/k = \nu/\alpha$
q	local heat-transfer rate per unit area
R	recovery factor
T	static temperature
ΔT	temperature difference; $T_w - T_{aw}$ with frictional heating, $T_w - T_\infty$ without frictional heating
$\dot{T}_w, \ddot{T}_w$	time derivatives of wall temperature

t time

U_∞ free-stream velocity

u velocity component in x-direction

v velocity component in y-direction

X,x coordinate measuring distance along plate from leading edge

Y transformed coordinate, $\int_0^y \frac{\rho}{\rho_\infty}\,dy$

y coordinate measuring distance normal to plate

α thermal diffusivity, $k/\rho c_p$

β_n expansion parameters, $\left(\frac{X}{U_\infty}\right)^n \frac{d^n T_w/dt^n}{T_w - T_\infty}$

γ ratio of specific heats

η Blasius similarity variable, $\frac{Y}{2}\sqrt{\frac{U_\infty}{\nu X}}$

θ dimensionless temperature, $(T - T_\infty)/(T_w - T_\infty)$

θ_1,θ_2 functions of η

μ absolute viscosity

ν kinematic viscosity

ρ density

τ time

Φ function of η

Ψ stream function

Subscripts:

aw adiabatic wall

inst instantaneous

qs quasi-steady

w wall

∞ free stream

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ANALYSIS

Governing Equations

The analysis begins with the equations expressing conservation of mass, momentum, and energy for unsteady laminar boundary-layer flow over a flat plate:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial \rho}{\partial t} = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

The boundary conditions appropriate to the problem are

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \\ T &= T_w(t) \end{aligned}\right\} y = 0 \qquad \left.\begin{aligned} u &\to U_\infty \\ T &\to T_\infty \end{aligned}\right\} y \to \infty \tag{4}$$

To simplify the treatment, the variation of fluid properties in liquids will be neglected, while for gases it will be supposed that the properties may be approximated by $\rho\mu$ = constant, ρk = constant, c_p = constant. Fluid-property variations are not expected to have a decisive effect on the final results (which are expressed in the form of ratios).

The equation for conservation of mass is satisfied by a stream function Ψ defined as follows (ref. 2, eq. (5)):

$$u = \frac{\rho_\infty}{\rho}\frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\rho_\infty}{\rho}\left(\frac{\partial \Psi}{\partial x} + \frac{\partial}{\partial t}\int_0^y \frac{\rho}{\rho_\infty}\,dy\right) \tag{1a}$$

Then, by replacing u and v in favor of Ψ and introducing the following new variables

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \qquad X = x, \qquad Y = \int_0^y \frac{\rho}{\rho_\infty}\,dy, \qquad \tau = t \tag{5}$$

equations (2) and (3) may be rephrased as

$$\Psi_{Y\tau} + \Psi_Y \Psi_{XY} - \Psi_X \Psi_{YY} = \nu_\infty \Psi_{YYY} \tag{2a}$$

$$\theta_\tau + \theta \frac{\dot{T}_w}{T_w - T_\infty} + \Psi_Y \theta_X - \Psi_X \theta_Y = \frac{\nu_\infty}{Pr}\left[\theta_{YY} + \frac{\mu_\infty/k_\infty}{T_w - T_\infty}(\Psi_{YY})^2\right] \tag{3a}$$

For brevity, the derivatives with respect to X, Y, and τ have been denoted by subscripts. In terms of the new variables, the boundary conditions (4) become

$$\left.\begin{aligned} \Psi_Y &= 0 \\ \Psi_X &= 0 \\ \theta &= 1 \end{aligned}\right\} Y = 0 \qquad\qquad \left.\begin{aligned} \Psi_Y &\rightarrow U_\infty \\ \theta &\rightarrow 0 \end{aligned}\right\} Y \rightarrow \infty \tag{4a}$$

Since the transformation has removed all thermal effects from the momentum equation (2a), it is clear that $\Psi_{Y\tau} \equiv 0$.

Solutions

As has already been noted, the object of this analysis is to investigate the deviations of the actual instantaneous heat transfer from the quasi-steady value. With this in mind, it is natural to seek a solution for the temperature distribution in the form of a series expansion about the quasi-steady state. In reference 1, the following quantities were introduced to serve as expansion parameters for the series solution:

$$\beta_1 = \frac{X}{U_\infty}\left(\frac{\dot{T}_w}{T_w - T_\infty}\right), \qquad \beta_2 = \left(\frac{X}{U_\infty}\right)^2\left(\frac{\ddot{T}_w}{T_w - T_\infty}\right), \quad . \; . \; . \tag{6}$$

Physically, these parameters represent the ratio of the time required for changes in surface temperature to diffuse across the boundary layer[1]

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[1]The time required for changes to diffuse across a layer of thickness Δ is proportional to Δ^2/α; while the thermal boundary-layer thickness Δ is proportional to $\sqrt{x\alpha/U_\infty}$. Hence, the diffusion time is proportional to x/U_∞.

to a time that is characteristic of the rapidity of the surface temperature changes. Thus, the β_n serve as a measure of the promptness with which the boundary layer responds to impressed variations of surface temperature.

Using the β_n, the series solution for the dimensionless temperature θ is written as

$$\theta = \theta_0(\eta) + \beta_1\theta_1(\eta) + \beta_2\theta_2(\eta) + \ldots + \frac{U_\infty^2}{2c_p(T_w - T_\infty)} \Phi(\eta) \tag{7}$$

where η is the well-known Blasius similarity variable given by

$$\eta = \frac{Y}{2X}\sqrt{\frac{U_\infty X}{\nu}} \tag{8}$$

The functions θ_0 and Φ are associated with the quasi-steady temperature distribution. When the β_n are small (corresponding to prompt response to impressed changes), the state is essentially quasi-steady.

The stream function Ψ is written in terms of the Blasius variable F as

$$\Psi = \sqrt{\nu_\infty U_\infty X}\, F(\eta) \tag{9}$$

When the expressions for θ and Ψ are substituted back into equations (2a) and (3a), and after terms are grouped according to whether they are multiplied by β_1, β_2, . . ., the following set of ordinary differential equations results:

$$F''' + FF'' = 0 \qquad F(0) = F'(0) = 0,\ F'(\infty) = 2 \tag{10}$$

$$\theta_0 + \mathrm{Pr}F\theta_0' = 0 \qquad \theta_0(0) = 1,\ \theta_0(\infty) = 0 \tag{11}$$

$$\theta_1'' + \mathrm{Pr}\left(F\theta_1' - 2F'\theta_1 - 4\theta_0\right) = 0 \qquad \theta_1(0) = \theta_1(\infty) = 0 \tag{12}$$

$$\theta_2'' + \mathrm{Pr}\left(F\theta_2' - 4F'\theta_2 - 4\theta_1\right) = 0 \qquad \theta_2(0) = \theta_2(\infty) = 0 \tag{13}$$

$$\Phi'' + \mathrm{Pr}\left[F\Phi' + \frac{1}{2}(F'')^2\right] = 0 \qquad \Phi(0) = \Phi(\infty) = 0 \tag{14}$$

A solution of equation (10) was first obtained by Blasius in 1908, but it was necessary to re-solve this equation to greater accuracy for present purposes. Numerical solutions to equations (11), (12), and (13) have been carried out as part of this investigation on an IBM 650 Magnetic Drum Data-Processing Machine for Prandtl numbers of 0.01, 0.72, 1.0, and 10. The function Φ is associated with the aerodynamic heating.

For laminar conditions, this effect of viscous dissipation is usually of importance only for high-speed gas flows. Solutions of the aerodynamic-heating problem for the Prandtl number range of gases are available in the literature (e.g., ref. 6).

These numerical results will be utilized in the heat-transfer calculations that follow.

HEAT-TRANSFER RESULTS

The instantaneous local heat flux at the plate surface q_{inst} may be calculated by applying Fourier's law:

$$q = -\left(k\frac{\partial T}{\partial y}\right)_{y=0}$$

After introducing the series expansion of (7) and taking account of the transformed variables of equations (5) and (8), the expression for q becomes

$$q_{inst} = -\frac{k}{2}\sqrt{\frac{U_\infty}{\nu x}}(T_w - T_\infty)\left[\theta_0'(0) + \beta_1\theta_1'(0) + \beta_2\theta_2'(0) + \ldots + \frac{U_\infty^2}{2c_p(T_w - T_\infty)}\Phi'(0)\right] \tag{15}$$

where $\theta_0'(0)$, $\theta_1'(0)$, . . . are abbreviations for $(d\theta_0/d\eta)_{\eta=0}$, ...

For liquids, the effects of aerodynamic heating will be neglected, and consequently the last term on the right side of the equation is omitted. For gases, it is convenient to introduce the adiabatic wall temperature T_{aw} by the relation

$$T_{aw} = T_\infty\left[1 + R\frac{(\gamma - 1)}{2}M_\infty^2\right] = T_\infty\left(1 + R\frac{U_\infty^2}{2c_pT_\infty}\right) \tag{16}$$

where R, the recovery factor, has been given in reference 1 as

$$R = -\frac{\Phi'(0)}{\theta_0'(0)} \tag{16a}$$

Making use of equations (16a), (16), and (6), it is easy to show that the final expression for q_{inst} has the same form for liquids and for gases, namely,

$$q_{inst} = -\frac{k}{2}\sqrt{\frac{U_\infty}{\nu x}}\,\Delta T\left[\theta_0'(0) + \frac{\dot{T}_w}{\Delta T}\left(\frac{x}{U_\infty}\right)\theta_1'(0) + \frac{\ddot{T}_w}{\Delta T}\left(\frac{x}{U_\infty}\right)^2 \theta_2'(0) + \ldots\right] \tag{17}$$

where

$$\Delta T = T_w - T_{aw} \qquad \text{for gases}$$

$$\Delta T = T_w - T_\infty \qquad \text{for liquids}$$

The quasi-steady heat transfer q_{qs} is given by

$$q_{qs} = -\frac{k}{2}\sqrt{\frac{U_\infty}{\nu x}}\,\Delta T\,\theta_0'(0) \tag{18}$$

The important relation between the instantaneous and the quasi-steady heat transfer is then found by combining equations (17) and (18):

$$\frac{q_{inst}}{q_{qs}} = 1 + \frac{x}{U_\infty}\left[\frac{\dot{T}_w}{\Delta T}\left(\frac{\theta_1'(0)}{\theta_0'(0)}\right) + \frac{\ddot{T}_w}{\Delta T}\left(\frac{x}{U_\infty}\right)\left(\frac{\theta_2'(0)}{\theta_0'(0)}\right) + \ldots\right] \tag{19}$$

The quantities $\theta_0'(0)$, $\theta_1'(0)$, and $\theta_2'(0)$ have been found as part of the solution of equations (11), (12), and (13) and are listed in table I. For convenience, the ratios $\theta_1'(0)/\theta_0'(0)$ and $\theta_2'(0)/\theta_0'(0)$ are plotted in figure 1 as a function of Prandtl number. It is seen from equation (19) that the deviations from quasi-steady heat transfer depend directly on the magnitudes of these ratios. Figure 1 shows that both ratios increase markedly with Prandtl number, demonstrating, for example, that liquid metals ($Pr \approx 0.01$) are less likely to experience deviations from quasi-steady heat transfer than is water ($Pr \approx 5$). This trend might have been intuitively expected on physical grounds as a consequence of the relatively high thermal diffusivity of liquid metals.

As alternative form of the results may be obtained by introducing heat-transfer coefficients as follows:

$$h_{inst} = \frac{q_{inst}}{\Delta T}, \qquad h_{qs} = \frac{q_{qs}}{\Delta T} \tag{20}$$

Then, equation (19) may be rephrased as

$$\frac{h_{inst}}{h_{qs}} = 1 + \frac{x}{U_\infty}\left[\frac{\dot{T}_w}{\Delta T}\left(\frac{\theta_1'(0)}{\theta_0'(0)}\right) + \frac{\ddot{T}_w}{\Delta T}\left(\frac{x}{U_\infty}\right)\left(\frac{\theta_2'(0)}{\theta_0'(0)}\right) + \ldots\right] \tag{21}$$

Equation (21) is in a form useful for interpretation of heat-transfer coefficients obtained under transient conditions.

4868 To further facilitate calculations, a plot of the quasi-steady heat-transfer relation (eq. (18)) is presented in figure 2.

Criterion for Quasi-Steady Heat Transfer

From the series nature of the solution, it is expected that equation (19) would be most accurate for small deviations of q_{inst}/q_{qs} from unity. This suggests that this equation can serve as an accurate and rapid means for checking whether a given situation may be treated as quasi-steady. When the free-stream velocity and surface temperature data of a particular situation lead to $q_{inst}/q_{qs} \approx 1$, then that situation can be taken as quasi-steady for heat-transfer purposes.

In a great many forced-convection applications, the quotient x/U_∞ (in seconds) will be a small number. As a consequence, the second term in the brackets of equation (19) will often be very small compared with the first term. Under these circumstances, it is possible to establish a simple criterion for determining when heat transfer from a surface with time-dependent temperature can be computed with sufficient accuracy from quasi-steady relations. Suppose that it is decided that an accuracy of 5 percent is sufficient for many applications. Then, the upper curve of figure 3 distinguishes the conditions under which deviations from the quasi-steady state may be ignored in the computation of local heat transfer. Alternatively, if an accuracy of 2 percent is required, the lower curve of figure 3 gives the conditions under which quasi-steady relations are adequate. As a simple example, suppose that water (Pr = 5) is flowing at 50 feet per second over a flat plate. If $T_w - T_\infty = 100^\circ$ F and $x = 1$ foot, then the departures from the quasi-steady state will have negligible effect (≤ 2 percent) on the local heat transfer when the rate of change of surface temperature does not exceed 23° F per second at $x = 1$ foot.

CONCLUDING REMARKS

Although the analysis given here is for laminar flow over a flat plate, the findings may have a wider utility. Since the response of a turbulent flow should be more rapid than that of laminar flow, the present results should give an upper bound on the deviations from turbulent quasi-steady heat transfer. Further, it is expected that the results reported here will apply approximately in the entrance region of ducts.

It is felt that the final results should not be strongly sensitive to fluid-property variations, although caution should be exercised when fluids with unusual property variations are involved (e.g., near the critical point).

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, March 28, 1958

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TABLE I. - FUNCTIONS NEEDED IN HEAT-

TRANSFER COMPUTATIONS

Pr	$\theta_0'(0)$	$\theta_1'(0)$	$\theta_2'(0)$	$\frac{\theta_1'(0)}{\theta_0'(0)}$	$\frac{\theta_2'(0)}{\theta_0'(0)}$
0.01	-0.1032	-0.1169	0.02034	1.133	-0.1971
.72	-.5913	-1.416	.4739	2.395	-.8015
1.0	-.6641	-1.751	.6453	2.637	-.9717
10	-1.456	-7.993	6.140	5.489	-4.216

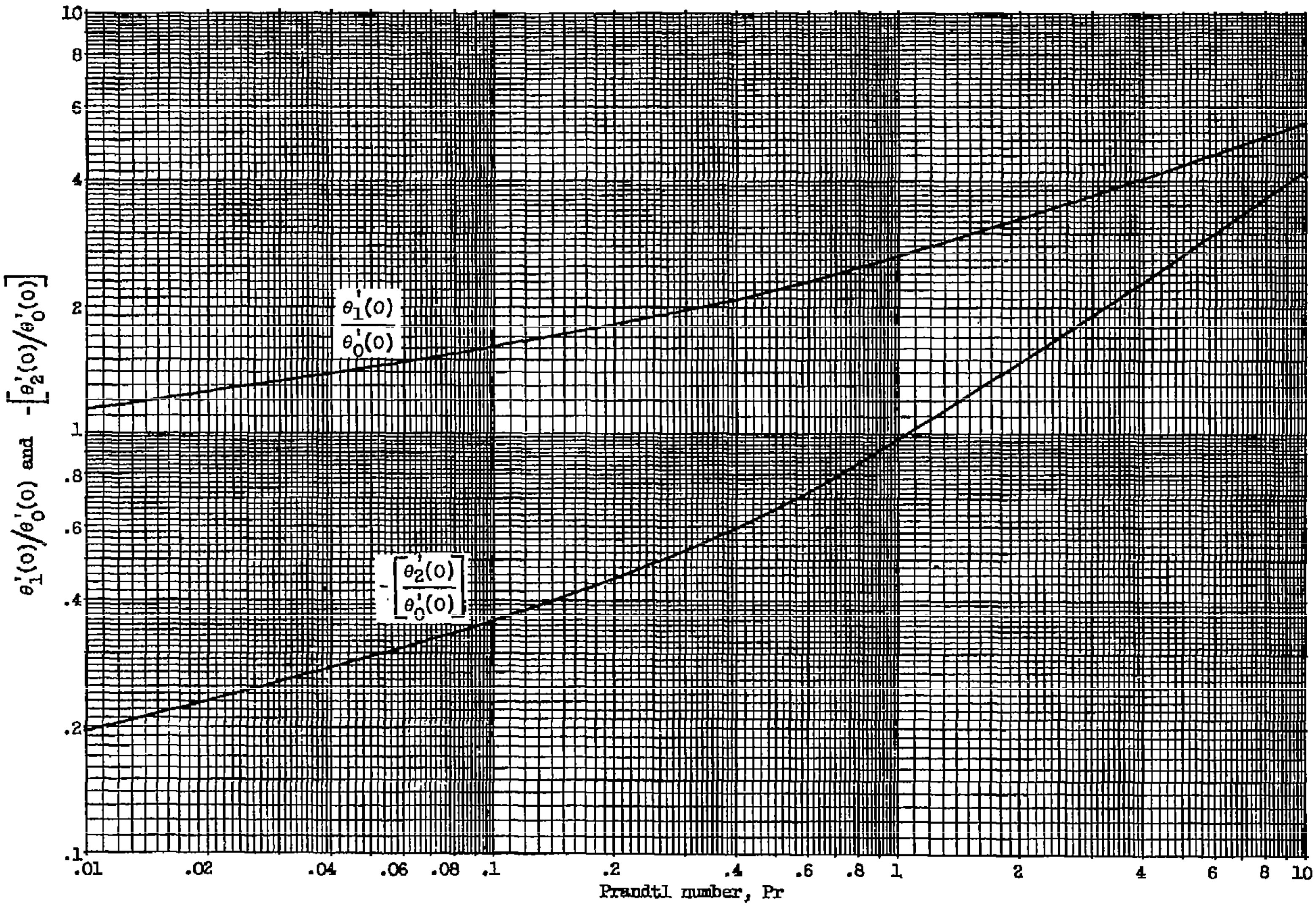

Figure 1. - Functions needed for computing heat transfer in equation (19).

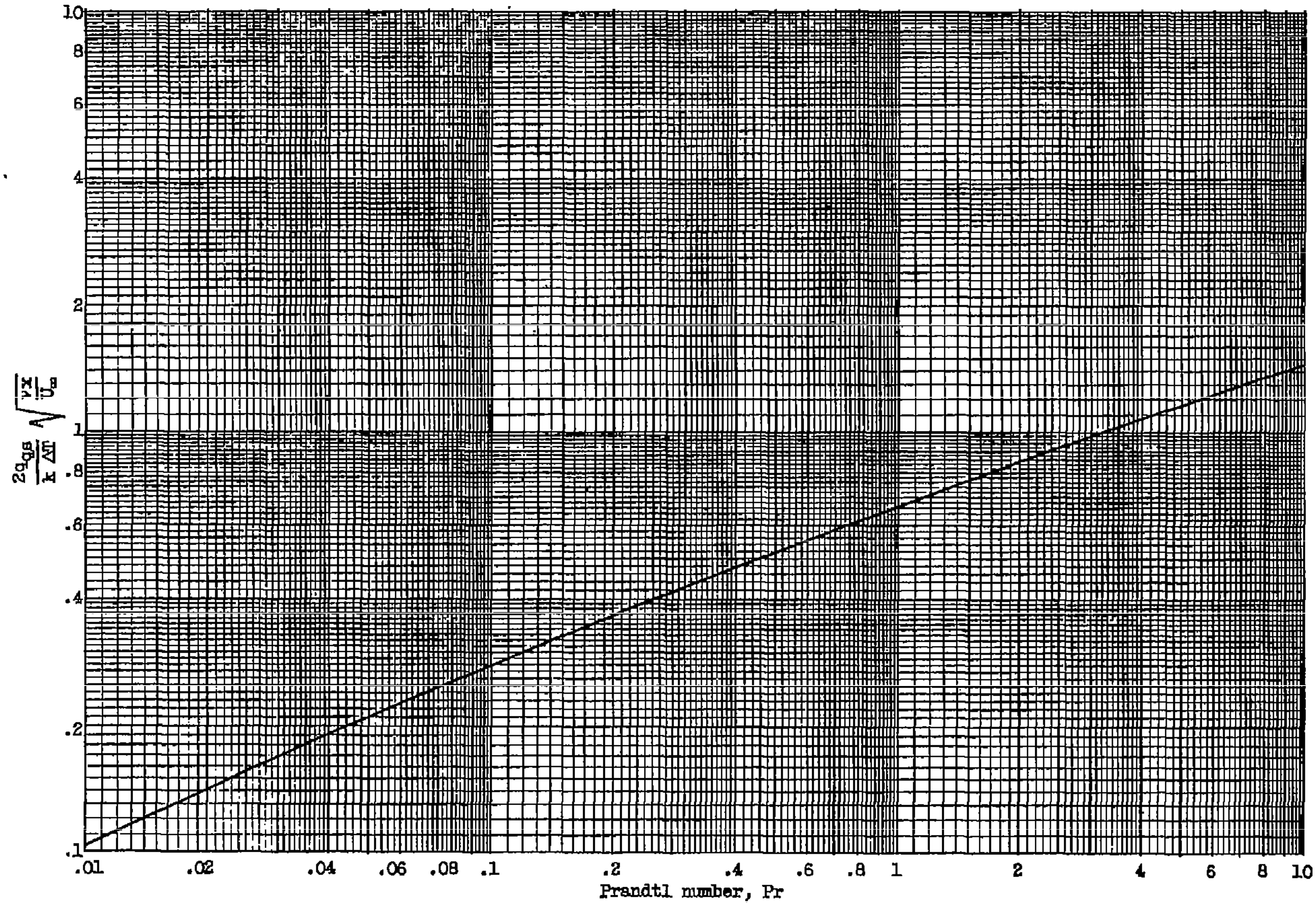

Figure 2. - Quasi-steady heat-transfer results.

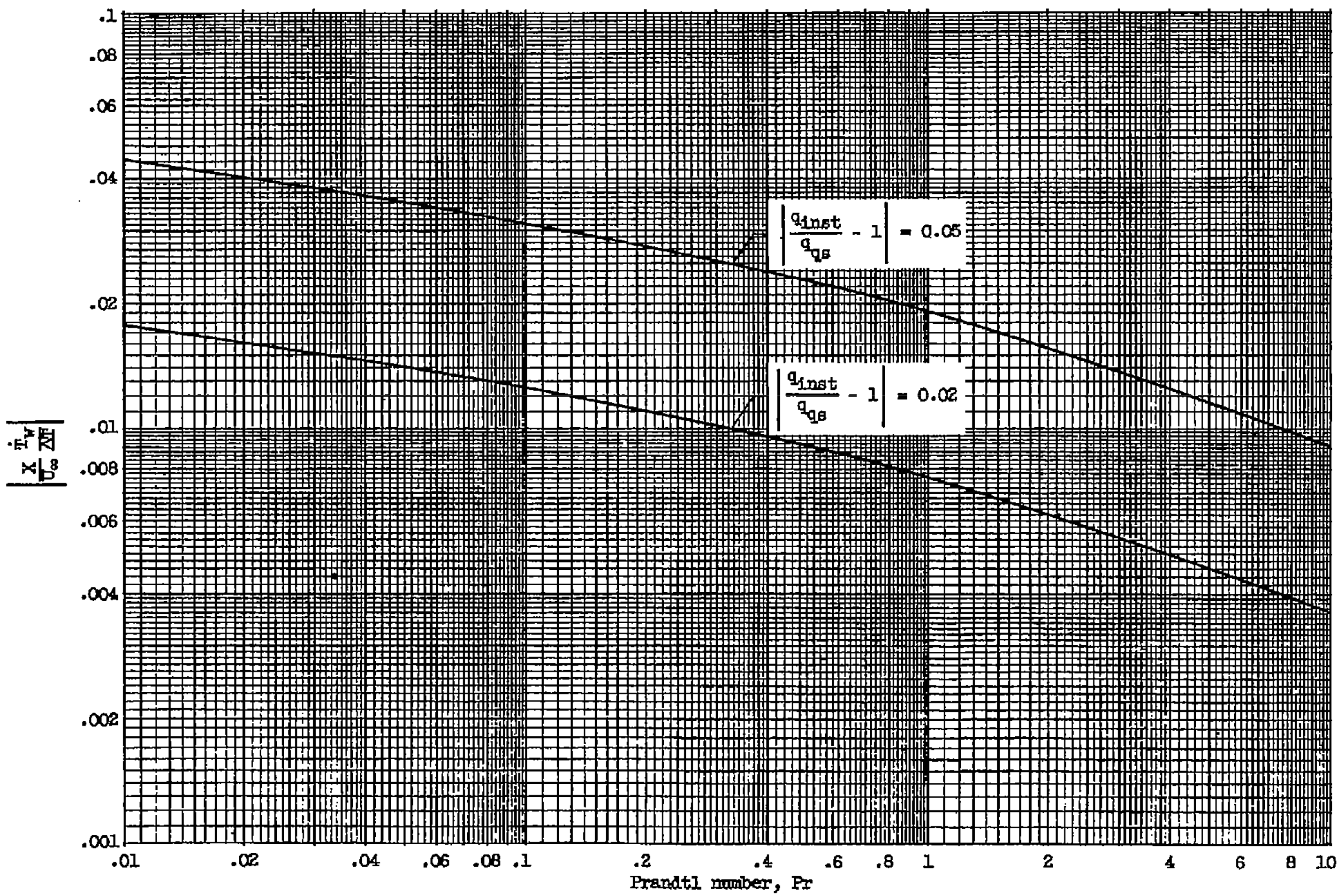

Figure 3. - Values of $\left|\frac{X}{U_\infty}\frac{\dot{T}_w}{\Delta T}\right|$ corresponding to 2- and 5-percent deviations of q_{inst} from q_{qs} (based on first term of eq. (19)).